Answers

The sheet with its answers, **in blue**. Question 1 is a guess and has no wrong answer; every other question has one right one, except 8(c), where the chain depends on a tie you are free to break either way.

Part 1 — A letter from Omaha

Question 1: The letter arrived. How many hands did it pass through on the way? Write your guess, and write why you guessed that.

⇒ A guess, and the room's guesses are the material for the session. The number is in 8(d): the chains that arrived ran about **five or six** people long. Most guesses in a lecture hall are far too high, which is exactly what makes the arithmetic in 8(a) worth doing.

Part 2 — Ringville

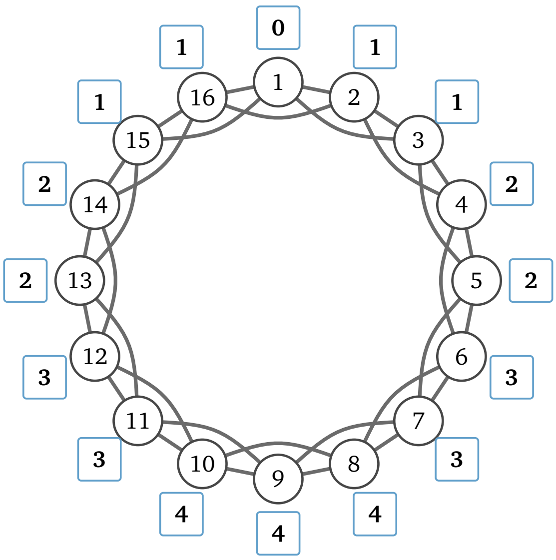
Sixteen people live in Ringville and they sit in one circle. Everybody is friends with the two on their left and the two on their right: four friends each, thirty-two friendships in the town.

Question 2(a): Person 1 has a letter for person 9. Trace the chain on the drawing. How many handshakes does it take?

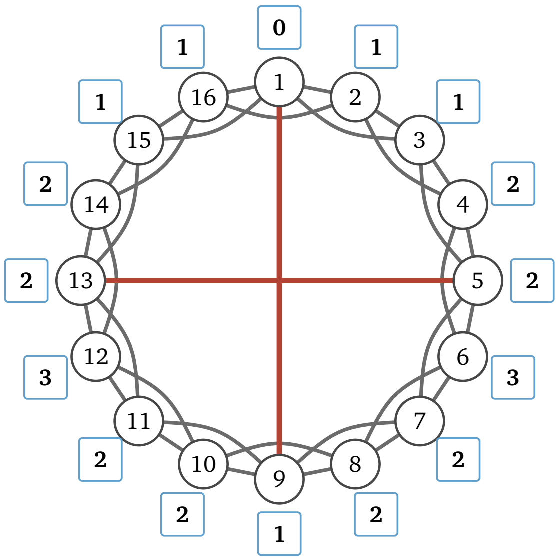
⇒ **Four**, and there are exactly two chains that do it: **1–3–5–7–9** and **1–15–13–11–9**. Person 9 is eight chairs away and no single handshake covers more than two chairs, so nothing beats four.

Question 2(b): Now everybody. In each person's box on the drawing, write how many handshakes it takes to reach them from person 1. Two are written in already, and you may find you only have to work out half of the rest.

⇒ Below left. Going round one way gives the same numbers as going round the other, so people 10 to 16 are people 8 to 2 read backwards — half the work, and the reason the drawing is symmetric is that person 1 has no idea which way is which.



2(b): Ringville



4: with the two shortcuts

Question 2(c): Add up every number in the boxes. What is the total? Share that total out among the fifteen other people — the town's average. And what is the biggest number in any box, the town's worst case?

⇒ Total **36**, average $36/15 = \mathbf{2.4}$, worst case **4**, which three people share: 8, 9 and 10.

Question 2(d): Take the average and the worst case. One of the two you would give to a newspaper; the other you would give to an engineer who has to promise the letter arrives. Which is which, and why?

⇒ **2.4 is the newspaper's** number and **4 is the engineer's**. An average describes the letter you would typically send; it promises nothing about any particular one. The worst case is the only number you can put a guarantee behind, because no letter in this town ever takes longer. Both get names in the lecture: **average path length** and **diameter**.

Question 3: Ringville grows. A thousand people, still one circle, still four friends each. How many handshakes does it take to reach the person sitting directly opposite you? And if the circle held all eight billion people on Earth? Could a world wired like Ringville get the Omaha letter to Boston in six handshakes?

⇒ With a thousand people the person opposite is 500 chairs away, so **250** handshakes. With eight billion they are four billion chairs away, so **two billion**. **No** — in a ring the distance grows in step with the town, so a ring-wired world would need billions of handshakes, not six. Whatever explains Milgram, it is not this shape.

Part 3 — Two shortcuts

Two of the sixteen went to college and came home with a friend from the far side of town: person 1 is now also friends with person 9, and person 5 with person 13 — thirty-four friendships, not thirty-two.

Question 4: Write the handshake counts into the boxes again, this time for the town with the two new friendships. Then the same three numbers as in 2(c): the new total, the new average, and the new worst case.

⇒ Above right. Total **27**, average $27/15 =$ **1.8**, worst case **3**, now reached only by persons 6 and 12.

Question 5: Thirty-two friendships became thirty-four. Work out both changes as percentages: by how much did the number of friendships change, and by how much did the average change? Say what those two new friendships did that thirty-two ordinary ones could not.

⇒ Friendships up by $2/32 =$ **6.25%**; the average down from 2.4 to 1.8, which is **25%**. Four times the effect, for one sixteenth of the wiring.

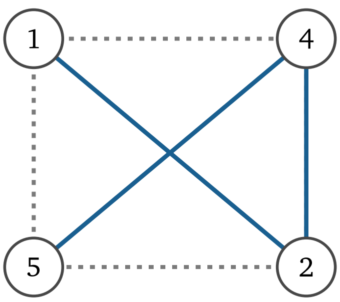
An ordinary Ringville friendship joins two people who were already one or two handshakes apart, so it shortens nothing for anybody. These two joined people *four* handshakes apart — and the saving is not theirs alone. Once person 1 can reach person 9 in a single step, so can everybody sitting near person 1, for everybody sitting near person 9. One line rewrote the distance between two whole neighbourhoods.

Part 4 — Do your friends know each other?

Go back to Ringville as it was in Part 2, before the two shortcuts, and take somebody ordinary. Person 3's four friends are persons 1, 2, 4 and 5, and those four make six pairs.

Question 6(a): Check the six pairs against the circle in Part 2, one pair at a time. Darken the dashed lines that are real friendships. How many of the six are?

⇒ **Three of six**, drawn below: **1 & 2**, **2 & 4** and **4 & 5**. The other three — 1 & 4, 1 & 5, 2 & 5 — sit three or four chairs apart, which is further than a Ringville friendship reaches.



Question 6(b): Ringville grows to ten thousand people, still four friends each. How many of person

3's six pairs are friends now?

⇒ **Still three of six.** Person 3's friends are still the people in chairs 1, 2, 4 and 5, and whether two of them are friends depends only on how many chairs apart they sit. The size of the town never enters the question.

Question 6(c): A different town: the same sixteen people and the same thirty-two friendships, but this time drawn out of a hat. Pick two people. What is the chance that they are friends? So how many of person 3's six pairs would you expect to be friends? And in a hat-drawn town of ten thousand, still four friends each?

⇒ Each person has 4 friends among the 15 others, so the chance is $4/15 \approx$ **0.27**, and $6 \times 4/15 =$ **1.6** of the six pairs. In a hat-drawn town of ten thousand it is $4/9999$, about one in 2,500, so $6 \times 4/9999 =$ **0.0024 — effectively none**.

Question 6(d): That fraction has a name — it is person 3's **local clustering**. You have now measured it four times. Say in one sentence what makes it big and what makes it small.

⇒ It is **big when your friends are drawn from the same small pocket of the world you are drawn from**, because then they are in each other's pockets too, and **small when they are drawn from the whole town**. Ringville holds $1/2$ at any size; the hat-drawn town falls away as $1/n$. That is the whole reason the two halves of this module are a puzzle at all: **short paths come free from randomness, and high clustering does not**.

Three people who are all friends with each other make a **triangle**. Ringville, before the two shortcuts, has sixteen.

Question 7(a): Then the two shortcuts went in. Do persons 1 and 9 have a friend in common? Do persons 5 and 13? So how many triangles does the town have now?

⇒ Person 1's friends are 15, 16, 2, 3; person 9's are 7, 8, 10, 11 — **no overlap**, and **none** for 5 and 13 either, so neither new line closes a triangle. And a line that is *added* can never destroy one. So **sixteen, unchanged**. The average fell by a quarter and the town lost nothing.

Question 7(b): The real model does not *add* a shortcut, it **moves** one: person 1 drops person 2 and befriends person 9 instead. Which friends did persons 1 and 2 have in common? So how many triangles does the town lose?

⇒ Persons 1 and 2 both know **person 16 and person 3**, so cutting 1–2 kills the triangles **1–2–16** and **1–2–3**: **two** go, and the town is left with **fourteen**. This is what the Watts–Strogatz model actually does at each step, and it is the honest version of 7(a).

Question 7(c): Put Question 5 beside 7(a) and 7(b). What does a shortcut buy the town, and what does it cost?

⇒ It buys **a quarter off the average distance** for two lines out of thirty-four. It costs **nothing when the line is added, and two triangles out of sixteen when it is moved** — and the triangles it breaks are all local to the one person it was moved from. **Distance is cheap to buy and clustering is expensive to lose**, so a handful of shortcuts collapses the distances and leaves almost all of the local structure standing. That is the mechanism the whole model rests on.

Part 5 — Back to Omaha

Question 8(a): Everybody knows 150 people. Pretend for a moment that no two of your friends know each other, so every handshake reaches 150 people nobody in the chain has met yet. Carry the table two rows further. At how many handshakes do you pass eight billion?

handshakes	people reached
1	150
2	22,500
3	about 3.4 million
4	**about 506 million**
5	**about 76 billion**

⇒ **Five.** Four handshakes reach half a billion, which is not enough; five overshoot the planet by a factor of nine. Six degrees was never a measurement, it was this multiplication.

Question 8(b): In Question 6 you found that person 3's friends do know each other, so the pretending in 8(a) is false. Which way does that push the answer — more handshakes, or fewer? Why?

⇒ **More handshakes.** If your friends know each other — and Question 6 says half their pairs do — then most of the 150 people your friend reaches are people you were already reaching yourself. Every step adds far fewer than 150 times as many new people, so the tower of powers above is an **overestimate of the reach** and therefore an **underestimate of the number of handshakes**. Clustering is exactly what makes six degrees hard.

Question 8(c): You are person 3 in the town with the two shortcuts, holding a letter for person 12. You know only your own four friends and the chair each of them sits in, and you hand it to whichever of them sits closest to chair 12; they do the same. Write the chain that rule produces, then the shortest chain that exists.

⇒ Person 3's friends sit in chairs 1, 2, 4 and 5, which are 5, 6, 8 and 7 chairs from chair 12 — so the letter goes to **person 1**. Person 1's friends are 15, 16, 2, 3 and 9, and **15 and 9 tie** at three chairs away. Either way it takes four:

3 — 1 — 15 — 13 — 12 or **3 — 1 — 9 — 11 — 12** **4 handshakes**

The shortest chain is **3 — 5 — 13 — 12**, **3 handshakes**.

Person 3 never sends it that way, and is not being stupid. Person 5 sits *seven* chairs from the target — further round than person 1 — so by the only rule person 3 can apply, person 5 is the wrong way. What person 3 cannot know is that person 5 is the one holding the door to 13.

Question 8(d): Of 296 letters sent from Omaha, 64 arrived, in five or six steps. Is "a short chain exists" the same claim as "an ordinary person can find one"? Which of the two did the letters test, and what do the 232 that never arrived say about it?

⇒ **Not the same claim**, and 8(c) is a town where the second is false while the first is true. The letters tested **the finding, not the existing**: nobody in the chain ever saw the network, and each of them made one local guess.

And **232 of the 296 never arrived at all**. A chain that stalls is not evidence that no short chain existed — it is evidence that somebody could not see it, or could not be bothered. The experiment measured the length of the chains that *worked*, which is a number about people, not only about the network. That distinction is what the lecture is for.

Part 6 — The lab

Three functions, and everything else in the notebook runs off them.

```
def ring_edges(n, half):
    return [(i, (i + d) % n) for i in range(n) for d in range(1, half + 1)]

def distances_from(A, s):
    n = len(A)
    dist = np.full(n, -1)
    dist[s] = 0
    frontier = [s]
    while frontier:
        nxt = []
        for u in frontier:
            for v in np.flatnonzero(A[u]):
                if dist[v] < 0:
                    dist[v] = dist[u] + 1
                    nxt.append(v)
        frontier = nxt
    return dist

def local_clustering(A, i):
    nbrs = np.flatnonzero(A[i])
    k = len(nbrs)
    if k < 2:
        return 0.0
    links = A[np.ix_(nbrs, nbrs)].sum() / 2
    return links / (k * (k - 1) / 2)
```

Run on the sheet's town they give **36 / 2.4 / 4** and **0.5** for every person, which are Questions 2(c) and 6(a) again, out of code that never looked at the drawing.

Finished early — the number that lies. The notebook then measures $\sigma = (C/C_{\text{rand}})/(L/L_{\text{rand}})$ on a plain ring of *one thousand* people with **no shortcuts at all**, and gets $\sigma = 4.96$: a strong small world, by a test applied to a town that has none. Turn the n dial up and the verdict gets *stronger*, not weaker.

The four quantities, for a ring of n people with k friends each:

$$C = \tfrac{1}{2}, \qquad L \approx \tfrac{n}{8}, \qquad C_{\text{rand}} = \tfrac{k}{n}, \qquad L_{\text{rand}} \approx \tfrac{\ln n}{\ln k}$$

$$\frac{C}{C_{\text{rand}}} = \frac{n}{8}, \qquad \frac{L}{L_{\text{rand}}} = \frac{n}{8} \cdot \frac{\ln k}{\ln n}, \qquad \sigma = \frac{n/8}{(n/8)(\ln k/\ln n)} = \frac{\ln n}{\ln k}$$

The n cancels exactly. A ring is $n/8$ times more clustered than a random town *and* $n/8$ times further across, and σ divides one by the other, so it has measured nothing about this town at all. What survives the cancellation is the leftover $\ln n/\ln k$, which is there only because a random town's paths grow like $\ln n$ — so slowly that the ring's enormous path-length penalty is discounted by that factor, and the clustering side wins by it. Any network whose clustering does not fall with n scores $\sigma > 1$ automatically, shortcuts or no shortcuts. **σ was certifying that C is constant, not that shortcuts exist.**

The published repairs both add a *second* yardstick — a lattice, at the other end from the random graph — because "small world" was always a claim about the middle of a range, and σ only ever looked at one end:

$$\omega = L_{\text{rand}}/L - C/C_{\text{latt}} \qquad \text{(Telesford 2011; } -1 \text{ lattice, } 0 \text{ small world, } +1 \text{ random)}$$

At $n = 1000$, $k = 4$:

town	σ	ω	**SWI**
plain ring, $p = 0$	**4.96** passes	-0.96	0.00
rewired, $p = 0.05$	47.7	-0.41	**0.81** largest
rewired, $p = 0.2$	48.1	0.21	0.51
random, $p = 1$	0.47	0.93	0.00

The same network is a *strong small world* to σ and an *almost perfect lattice* to ω, and as n grows the two run in opposite directions. The price of the repair is that C_{latt} and L_{latt} have to come from somewhere: here the town *is* a lattice so they are free, and on real data you have to build a lattice with the same degrees yourself, which is not a small job. That is part of why σ is still the one people quote.

The whole notebook with every blank filled in is `lab-solutions.py`, built from `lab.py` by `tools/build_lab_notebooks.py m02-small-world`.