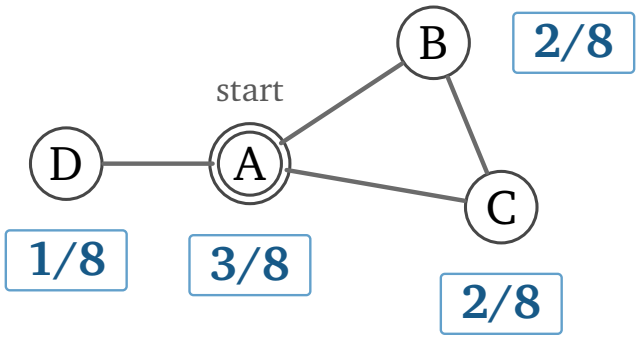
Answers

The sheet with its answers, **in blue**. Parts 1 and 2 are done before the lecture, Part 3 after it.

Part 1 — A walker with no map

Four friends: A, B, C and D. A walker starts at A and at every step moves to one friend of the person it is standing on, each with the same chance.



the boxes are filled in Question 6

Question 1: Before you compute anything, guess. After a very long walk, at which of the four people is the walker found most often? Why?

⇒ Any guess is fine; Question 6 checks it. The answer is **A**, the person with the most friends. A common wrong guess is D, because every walk that reaches D has to come straight back through A.

Question 2: In each cell of the table, write the chance that the walker moves from the person of that row to the person of that column in one step. Then add up each row.

from \ to	**A**	**B**	**C**	**D**	sum
A	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1
B	$\frac{1}{2}$	0	$\frac{1}{2}$	0	1
C	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
D	1	0	0	0	1

⇒ Each row is one over the number of friends of that row's person, in the columns of those friends. **Every row adds up to 1**: the walker has to go somewhere. This table is the matrix P of the lecture (slide 14), one row per starting person.

Question 3: The walker starts at A. What is the chance that it is at B after exactly three steps?

route	chance
$A \to D \to A \to B$	$\frac{1}{3} \times 1 \times \frac{1}{3} = \frac{1}{9}$
$A \to B \to A \to B$	$\frac{1}{3} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{18}$
$A \to C \to A \to B$	$\frac{1}{3} \times \frac{1}{2} \times \frac{1}{3} = \frac{1}{18}$
$A \to B \to C \to B$	$\frac{1}{3} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{12}$

$\Rightarrow$ Four routes, and the last row stays empty. $\frac{1}{9} + \frac{1}{18} + \frac{1}{18} + \frac{1}{12} = \frac{4+2+2+3}{36} = \frac{11}{36} \approx 0.31$. The routes that go back to A are the ones people miss: the walker has no memory, so it may undo its own step.

Part 2 — Drunk at the karate club

A university karate club in the 1970s, with 34 members and 78 friendships. The simulator (`go.skojaku.com/m07sim`) runs the same walker on it.

Question 4: Press **Run 5,000 steps** four times: 20,000 steps. Write the most and the least visited node in the table, with its edges and its share. Then divide the edges by the share.

	node	edges	share	edges ÷ share
most visits	33	17	0.109	156
fewest visits	11	1	0.0064	156

$\Rightarrow$ These are the long-run values. One run of 20,000 steps gives numbers near them, not equal to them: the ratio for node 33 usually lands between about 145 and 165, and for node 11 it can be 10% off. The most visited node is node 33 in about three runs out of four and node 0 (16 edges, share 0.103) in the others. Both give 156.

Question 5: Are the two numbers in the last column the same? How are they related to the 78 friendships?

$\Rightarrow$ **Nearly the same, and the same for everybody in the room: about 156 = 2 $\times$ 78.** The gaps between neighbours are the luck of one run, and they

shrink if you run longer. So the share of time at a node is its number of friends divided by twice the number of friendships:

$$\text{share} = \frac{\text{edges}}{2 \times \text{friendships}} = \frac{k}{2m}.$$

It is twice, not once, because every friendship has two ends, and the walker can arrive through either one.

Question 6: The four friends in Part 1 have 4 friendships. Work out the share of time at A, B, C and D. Was your guess in Question 1 right?

$\Rightarrow 2m = 8$: **A** $= \frac{3}{8}$, **B** $= \frac{2}{8}$, **C** $= \frac{2}{8}$, **D** $= \frac{1}{8}$ (the boxes on the drawing at the top). A gets most because it has the most friends. D gets least even though every visit to D is followed by a visit to A: D has one friend, and one friend is all that counts.

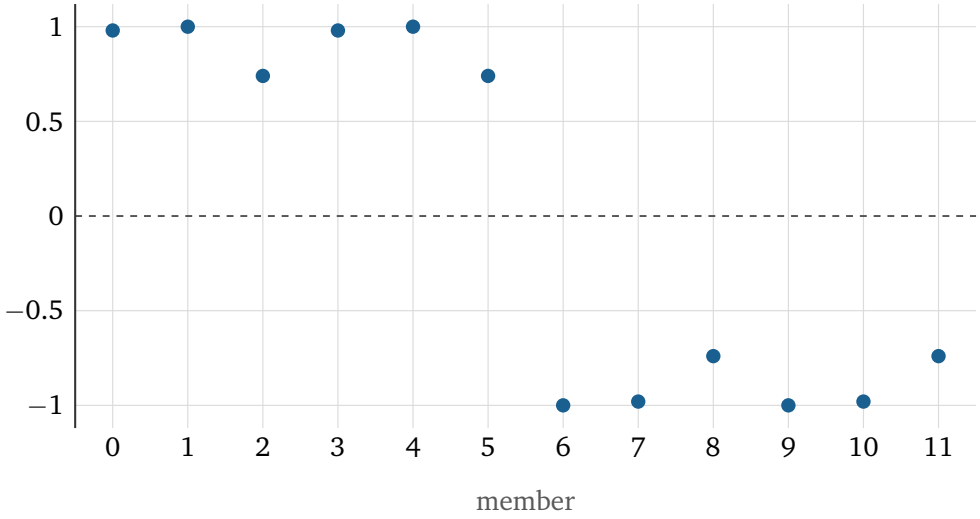
Part 3 — Two rooms

Twelve members and 30 friendships. The notebook (`go.skojaku.com/m07lab`) runs the cell printed on the sheet and prints

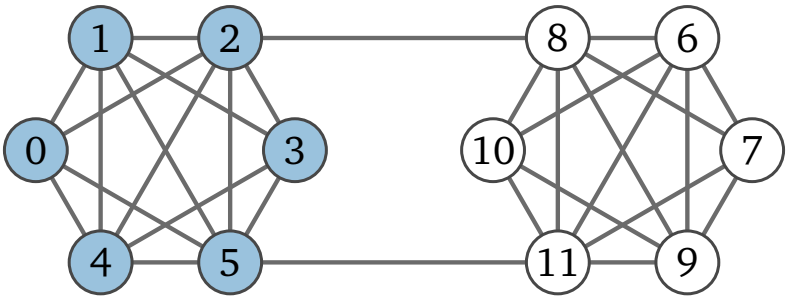
```
eigenvalues:        [1, 0.889, 0, 0, 0, -0.088, -0.2, -0.2, -0.266, -0.333, -0.379, -0.
second eigenvector: [0.98, 1, 0.74, 0.98, 1, 0.74, -1, -0.98, -0.74, -1, -0.98, -0.74]
```

(shortened here). On some machines every sign of the second eigenvector is the other way round. That is fine: an eigenvector times -1 is still an eigenvector, and the answers below only swap the two groups.

Question 7: Mark the twelve numbers of the second eigenvector as dots on the axes.



Question 8: On the drawing, darken every member whose number is above zero. What do the darkened members have in common?



⇒ **Members 0 to 5: one of the two groups.** Inside each group almost everybody is friends with everybody, and only two friendships cross between them. The sign of the second eigenvector names the group, which is why it is called a community signal (slide 44).

Question 9: Which four members have the numbers closest to zero? Where are they on the drawing?

⇒ **Members 2, 5, 8 and 11 (± 0.74): the ends of the two friendships that cross between the groups.** A walker at one of them is a single step from the other group, so it belongs to its own group a little less firmly.

Question 10: Work out 0.889^5, 0.889^{40} and 0.423^5. After 5 steps, which eigenvalues still have a term in the walk? And after 40 steps?

⇒ **$0.889^5 \approx 0.56$, $0.889^{40} \approx 0.009$, $0.423^5 \approx 0.013$.** Every other eigenvalue is smaller in size than 0.423, so its term is even smaller. **After 5 steps two terms are left: the eigenvalue 1, which never shrinks, and 0.889. After 40 steps only the eigenvalue 1 is left.**

Question 11: The lecture asked: short walks show communities, long walks show degree. Why? Answer in two sentences.

⇒ **Long walk: every term except the eigenvalue 1 has died, and that term is the share of time from Part 2, edges ÷ 2m, so all that is left is degree. Short walk: the second term, 0.889^t, is still large for the first ten or so steps, and its eigenvector has one sign in each group (Part 3), so the walker still remembers which group it started in.** The ten steps are the relaxation time of the lecture, $\tau = 1/(1 - 0.889) = 9$.

Question 12 (finished early): Delete the friendship 2–8 and run the cell again. Is the second eigenvalue closer to 1, or further from 1? Does a walker that starts at member 0 stay among members 0 to 5 for longer?

⇒ **Closer: 0.946. Longer.** The term 0.946^t shrinks more slowly than 0.889^t (0.76 against 0.56 after 5 steps), so the walker remembers its group for longer: with one way across instead of two, it takes longer to find the way out. The relaxation time grows from 9 to $1/(1 - 0.946) \approx 19$ steps. This is the bottleneck of slides 34–37.